How to use ICRFS-Plus^{TM} to fulfil Solvency II requirements?

Abstract

The European Parliament's Solvency II Directive, scheduled to come into effect on 1 January 2016, introduces new regulation for insurance. This aims to establish a consistently improved level of policyholder protection via a three-pillared process. The first pillar contains quantitative requirements for the insurance industry relating to Technical Provisions and the Solvency Capital Requirement. The reserve risk is a substantial contributor to the insurance risk and is addressed by the quantitative requirements.

We demonstrate ways that ICRFS-Plus^{TM} can be used to fulfil the quantitative requirements for non-life reserve risk in particular, in the context of the European Commission's Quantitative Impact Study (2010). This includes a standard formula with undertaking specific parameters and a partial Solvency II Internal Model.

Keywords Case Development Result; One-Year Risk Horizon; Solvency II Standard Formula; Solvency II Internal Model; Solvency Capital Requirement

1 Introduction

Insolvency risk is an inherent part of insurance business. Insurance risk, and reserve risk in particular is not hedgeable. The price of this risk cannot be determined by the market as there is no open market for insurance liabilities. For Solvency II Quantitative Requirements (European Commission et al. (2007a, 2009)), the risk is characterised by

- the risk profile: the distribution of basic own funds,

- the risk measure: value-at-risk, applied to Solvency Capital Requirements (SCR),
- risk tolerance: set at 99.5% with a one year time horizon (1-in-200-year distress event),
- risk margins: determined by using the Cost-of-Capital method.

The paper is organised as follows.

- We discuss important details from the Quantitative Impact Study.
- We describe the CDR, and how the standard deviation given ICRFS-Plus can be used with standard method 2.
- We discuss how the standard deviation from the aggregate of LoBs can be plugged into a partial internal model that extends standard method 2.
- We discuss a simulation-based internal model for non-life risk implemented in ICRFS-Plus, which is free from some of the limitations imposed by the CDR approach.
- A case study is presented that illustrates method 2 and the Internal Models.

2 The Quantitative Impact Study

The *Quantitative Impact Study*, QIS5 (European Commission et al. (2010)) addresses Quantitative Requirements in reserve risk for Solvency II by presenting three standard methods which can be used to estimate the standard deviation of the reserve risk for a line of business (LoB):

1. the *proportionality principle* - the variance of the best estimate for claims outstanding in one year plus the incremental claims paid over the same year is taken to be proportional to the current best estimate for claims outstanding (European Commission et al. (2010), SCR.10.40.)
2. an undertaking-specific standard deviation based on *a third party estimate* (European Commission et al. (2010), SCR.10.48.)

3. an undertaking-specific standard deviation based on *the Chain Ladder method* (European Commission et al. (2010), SCR.10.54.)

The standard methods 2 and 3 are known as Merz-Wüthrich approach (Merz and Wüthrich (2008)). We are sceptical about the proportionality principle which underlies method 1, see Munroe et al. (2015). We are also sceptical of the Chain Ladder method, see Barnett et al. (2005), Barnett and Zehnwirth (2008). Fortunately, method 2 allows for an undertaking-specific estimate of the standard deviation.

The standard deviation for reserve risk for each LoB is one of the source components for the QIS5 standard formula for SCR. The standard formula for SCR is invariant to the method that produces the standard deviation.

The Claims Development Result (CDR) concept is essential for standard methods 2 and 3 with undertaking-specific parameters, as the standard deviation of CDR quantifies the standard deviation of reserve risk.

Apart from the standard formulas, the regulation also allows for an Internal Model that estimates capital requirements for non-life risk, subject to supervisory approval. This model is only a *partial* internal model, as it covers non-life risk (reserve and premium risk) only.

3 Claims development result

The Claims development result (CDR) was introduced by Wüthrich et al. (2008) and is commonly used in the Solvency II literature. CDR is a random variable that represents the difference between the expected ultimate loss at inception (or at a given time, t) and one year later. It is commonly said that the $CDR(1)$ represents movement of an economic balance sheet between now and one year ahead.

Its distribution is linked with the Solvency Capital Requirement (SCR) that is nominally set to the 99.5th quantile of the basic own funds distribution limited to the year in question.

In fact, with two years in run-off, the CDR at inception could be defined as a function of L_1, the next year's loss, by the formula:

$$CDR(1) = E(L_1) - L_1 + E(L_2) - E(L_2|L_1),$$

where L_t is the loss in the future year t, and conditioning $L_2|L_1$ indicates that $CDR(1)$ is based on the joint distribution $f(L_1, L_2) = f(L_1)f(L_2|L_1)$.

This leads to (in-line with CDR-centric papers)

$$SCR^{cdr} = -VaR_{0.5\%}(CDR(1)) \tag{1}$$

which corresponds to

$$SCR^{cdr} = VaR_{99.5\%}(L_1) + E(L_2|L_1 = \lambda_1) - E(L_2),$$

where $VaR_{99.5\%}(L_1)$ is the 99.5% percentile of L_1 above its best estimate, $E(L_1)$, and

$$\lambda_1 = E(L_1) + VaR_{99.5\%}(L_1),$$

see Appendix B in Munroe et al. (2015) for technicalities.

Generalizing for n years in run-off is easy,

$$\begin{aligned} CDR(1) &= E(L_1) - L_1 + \sum_{t=2}^{n} (E(L_t) - E(L_t|L_1)) && (2)\\ &= E(L) - E(L|L_1), && (3)\\ SCR^{cdr} &= VaR_{99.5\%}(L_1) + \sum_{t=2}^{n} (E(L_t|L_1 = \lambda_1) - E(L_t)) && (4) \end{aligned}$$

where L is the ultimate loss

$$L = \sum_{t=1}^{n} L_t. \tag{5}$$

In the QIS5 standard formula for SCR, $\mathbf{S.D.(CDR(1))}$ *is the proxy for* **SCR** and (4) is not used, see Appendix A. SCR.9.16. in particular.

The CDR has zero mean. Taking expectation with respect to L_1 from both sides of (2),

$$E(CDR(1)) = 0. \tag{6}$$

The *variance of* $\mathbf{CDR(1)}$ *is essential to the Merz-Wüthrich approach.* Taking the variance of both sides of (3),

$$Var(CDR(1)) = Var(E(L|L_1)) \tag{7}$$

as $E(L)$ is a constant and $E(L|L_1)$ is a random variable.

By the law of total variance

$$Var(L) = E(Var(L|L_1)) + Var(E(L|L_1)). \tag{8}$$

Substituting (7) into above,

$$Var(L) = E(Var(L|L_1)) + Var(CDR(1)). \tag{9}$$

As $L|L_1$ is a random variable, $E(Var(L|L_1))$ is greater than zero, so the variance of $CDR(1)$ is less than the variance of the ultimate loss L

$$Var(CDR(1)) < Var(L). \tag{10}$$

The variance of the expected ultimate loss conditional on the first future year loss is less than the unconditional variance of the ultimate loss.

Generalizing of $CDR(1)$ and its properties for $CDR(t)$ is straightforward. In the following we focus on $CDR(1)$.

Calendar Yr Summary

Calendar Yr	Mean Outstanding	Standard Dev.	CV Outstanding	Cum. Means as % of total	Cond. on Next Cal. Per. √E[Var[Outs|Data]]	Cond. on Next Cal. Per. SD[E[Outs|Data]]
2009	1,446,622	92,276	0.06	62.92	0	92,276
2010	442,981	52,753	0.12	82.19	51,757	10,204
2011	219,811	48,728	0.22	91.75	48,131	7,602
2012	105,132	24,386	0.23	96.32	23,854	5,063
2013	49,508	12,254	0.25	98.48	11,865	3,064
2014	22,517	6,100	0.27	99.46	5,861	1,691
2015	9,421	2,923	0.31	99.87	2,805	822
2016	3,071	1,235	0.40	100.00	1,198	301
Total	2,299,063	137,806	0.06	100.00	82,832	110,134

1 Unit = 1,000 €

PCO S.D.(CDR)

Figure 1: ICRFS-Plus^{TM} conditional statistics

4 CDR standard deviation in ICRFS-Plus and QIS5 Standard Formula

The Probabilistic Trend Family (PTF) statistical modelling framework and its generalisation Multiple Probabilistic Trend Family (MPTF) (Barnett and Zehnwirth (2000), Zehnwirth et al. (2003)) are implemented in ICRFS-Plus. Conditional on the observed data, a PTF/MPTF model and a given forecast scenario, forecast summaries are produced. Importantly, both process variability and model parameter uncertainty are reflected in forecast summaries.

The conditional statistics displayed in the PTF and MPTF forecast summaries (see Figure 1) include the square root of the expectation of the conditional variance $\sqrt{E\left(Var\left(L_p|L_1\right)\right)}$, the standard deviation of the conditional expectation $S.D.\left(E(L_p|L_1)\right)$ for each future year, $\sqrt{E\left(Var\left(L|L_1\right)\right)}$ and $S.D.\left(E(L|L_1)\right)$ for the total. (By the law of total variance (8), the sum of squares of those columns is equal to the square of the unconditional standard deviation (column 2) for each row.)

From (7), $S.D.\left(E(L|L_1)\right)$ is the same as $S.D.(CDR(1))$. QIS5 refers to this quantity as $\sqrt{MSEP}$ in SCR.10.40. (See Appendix B for the mathematics behind $E\left(Var\left(L_p|L_1\right)\right)$.)

Method 2 in the QIS5 standard formula with undertaking-specific parameters in reserve risk allows the use of **S.D.(CDR(1))** and the mean (**PCO**, provision for claims outstanding) from a third party model. Those quantities can be obtained from the ICRFS-Plus reserve forecast summary, conditional statistics, total row, to derive undertaking-specific normalised LoB standard deviation for reserve risk $\sigma_{u,res,lob}$,

$$\sigma_{u,res,lob} = \frac{S.D.(CDR_{lob}(1))}{PCO_{lob}}. \tag{11}$$

$\sigma_{u,res,lob}$ is plugged into the *QIS5 SCR standard formula*, see figure 2.

The QIS5 SCR standard formula requires reserve and premium risk to be aggregated for each LoB into non-life risk, and then non-life risk to be aggregated between all LoBs, see Appendix A for details.

5 A partial ICRFS-Plus internal model for a LoB non-life risk

From the Solvency II Glossary (European Commission et al. (2007b), p.18),

> Premium risk only relates to future claims (excluding IBNR and IBNER), and originates from claim sizes being greater than expected, differences in timing of claims payments from expected, and differences in claims frequency from those expected.

From Gisler (2009) referred by QIS5 method 3 in premium risk with undertaking specific parameters (European Commission et al. (2010)),

> Premiums and administrative costs of next year can usually be forecasted with high accuracy and the risk involved in these two components are negligible.

Based on the above, premium risk can be addressed by the inclusion of a single future accident year in the future forecast summary in PTF.

It is straightforward to expand the mathematics for CDR from section 3 to a combined forecast in order to treat non-life risk in the same way as reserve risk. The definition of the loss L_t in a future calendar year t now includes the future loss attributed to the first future accident year. Thus, PTF/MPTF combined forecast summary facilitates non-life risk as a partial SII internal model.

Where the SCR standard formula predefines the correlation between reserve and premium risk as 0.5 (European Commission et al. (2010), SCR.9.31.), MPTF estimates this correlation, see combined forecast summary, Acc. Yrs tab, Correlations sub-tab.

The volume measure PCO and $S.D.(CDR(1))$ are available from the combined forecast summary, either Cal. Yrs Summary or Acc. Yrs Summary, total row, Mean Outstanding and $SD(E(Outs|Data))$ columns respectively.

Expanding method 2 with undertaking-specific parameters in reserve risk to non-life risk, the normalised LoB standard deviation

$$\sigma_{lob} = \frac{S.D.(CDR_{lob}(1))}{PCO_{lob}}.$$

This facilitates a reduced SCR chain, starting with SCR.9.32. (see Appendix A).

5.1 A note on volume measure and premium risk

If projected earned premiums in the first future accident year exceed the Total Mean Outstanding from the future forecast summary, then the volume measure can be increased by this difference, in line with SCR.10.13. from European Commission et al. (2010).

Although the above is sufficient to describe changes relating to a future accident year's earned premium risk, this does not allow for unexpired premium risk in the last accident year. This unexpired premium can be explicitly calculated by the insurer and could be typically assumed to represent a low proportion of the total premium risk. However, this may not always be the case, and the future exposure may need to be amended by not just the increase of projected earned premium vs outstanding, but also by the unexpired premium.

6 A partial ICRFS-Plus internal model for the aggregate of all LoBs non-life risk

The SCR standard formula predefines the correlation between LoBs (European Commission et al. (2010), SCR.9.32.).

MPTF modelling framework in ICRFS-Plus estimates common future trends and process correlations conditional on observed data and the selected model, see Insureware (2012), Zehnwirth (2014). The correlation between LoBs is displayed in combined forecast summary, Aggregate, Summary by Datasets, Correlations, Totals.

The PCO as a volume measure is available from combined forecast summary, Aggregate Cal. Yrs Summary, total row, Mean Outstanding column.
The $S.D.(CDR(1))$ is available from combined forecast summary, Aggregate Cal. Yrs Summary, total row, $SD(E(Outs|Data))$ column.

Expanding method 2 with undertaking-specific parameters in reserve risk to non-life risk, the normalised LoB standard deviation

$$\sigma = \frac{S.D.(CDR(1))}{PCO}.$$

This facilitates a further reduced SCR chain, eliminating SCR.9.32. (see Appendix A).

7 Simplifying approximations in the QIS5 SCR standard formula

This partial internal model presented above is free from the QIS5 assumptions about

- correlation between reserve and premiums risk inside LoB and
- correlation between LoBs.

However, there are three outstanding simplifications.

1. *The time value of money is ignored.* When the loss in the first future calendar year exceeds the mean loss, the technical provision for the outstanding years in run-off is supposed to be topped-up by the quantities $E(L_t|L_1) - E(L_t)$. However, the parts of the technical provision attributed to different future years are supposed to accumulate risk-free interest. The interest attributed to each future year depends on how far away the year is from the origin. This consideration is not reflected in (2) as there is no present value discounting in the formula. This issue could be addressed in the ICRFS-Plus framework by discount factors for future years available in PTF and MPTF Forecast Setup dialog.

2. *There is no adjustment for risk margins.* A healthy economic balance sheet allocates sufficient funds for all future years until run-off. Those funds cover both expected mean loss and risk margins (market value margins, MVM) forming the technical provision. When the loss in the first future calendar year exceeds the mean loss, not only the mean loss for future years is supposed to be topped up, but risk margins as well. Risk margins are left out of the scope of (2).

3. *Lognormal distribution assumption.* SCR.9.18 (see European Commission et al. (2010)) assumes a lognormal distribution of underlying risk (of $CDR(1)$) for the aggregate of lines of business.

8 A partial simulations based ICRFS-Plus internal model for non-life risk

The inherent simplifications in SCR^{cdr} are the motivation for a simulation-based internal model as implemented in the SII module in ICRFS-Plus, see Munroe et al. (2015), Insureware (2011). The SCR produced can be plugged in directly into SCR.9.7. as NL_{pr}, see Appendix A.

9 Case Study

The purpose of this case study is to compare SCR estimates between QIS5 standard formula Method 2 (with undertaking-specific parameters in Reserve Risk sourced from ICRFS-Plus) and the simulation based internal model in ICRFS-Plus.

To make results comparable, the following assumptions are made in the QIS5 SCR chain (see Appendix A),

- the business is in run-off, there is no premium risk,
- there is only a single *aggregate* line of business,
- capital requirements for non-life lapse risk and non-life catastrophe risk are ignored (re: SCR.9.7),
- market risk, default risk, life risk and health risk are ignored in the calculation of BSCR (re: SCR.1.31),
- the risk-free rate set in the SII Setup Dialog as zero, inflation and discount rates are not in use in the Forecast Setup Dialog.

The dataset from Merz and Wüthrich (2008) is below.

Table 1. Accident Years vs Development Years loss development array

w \ d	0	1	2	3	4	5	6	7	8
0	2, 202, 584	3, 210, 449	3, 468, 122	3, 545, 070	3, 621, 627	3, 644, 636	3, 669, 012	3, 674, 511	3, 678, 633
1	2, 350, 650	3, 553, 023	3, 783, 846	3, 840, 067	3, 865, 187	3, 878, 744	3, 898, 281	3, 902, 425	
2	2, 321, 885	3, 424, 190	3, 700, 876	3, 798, 198	3, 854, 755	3, 878, 993	3, 898, 825		
3	2, 171, 487	3, 165, 274	3, 395, 841	3, 466, 453	3, 515, 703	3, 548, 422			
4	2, 140, 328	3, 157, 079	3, 399, 262	3, 500, 520	3, 585, 812				
5	2, 290, 664	3, 338, 197	3, 550, 332	3, 641, 036					
6	2, 148, 216	3, 219, 775	3, 428, 335						
7	2, 143, 728	3, 158, 581							
8	2, 144, 738								

We transform this cumulative triangle to incremental, apply the PTF wizard and run default forecast for model M6. The Calendar Forecast Summary produced (see Figure 1) is the QIS5 undertaking-specific method 2 in the reserve risk standard formula estimate for $S.D.(CDR(1))$ and PCO.

The values $S.D.(CDR(1)) = 106,916$ and $PCO = 2,298,680$ are plugged into QIS5 SCR chain (Appendix A) producing $BSCR = 289,608$, see Appendix C for technicalities. The SII module from ICRFS-Plus produces simulated $SCR = 295,012$ for the same dataset, model and forecast. The results are summarised in the table below.

Table 2. Comparison Table

	Merz and Wüthrich (2008)	$LOB1-6$	$LOB4$
PCO (Mean reserve)	$2,298,680$	$659,862$	$56,456$
$S.D.(CDR)$	$106,916$	$29,524$	$14,956$
ρ (SCR.9.16)	0.126	0.121	0.891
$BSCR$ (SCR.1.31)	$289,608$	$79,822$	$50,283$
SCR (simulations)	$295,012$	$68,390$	$30,354$
$BSCR$ relative to SCR	**98.17%**	**116,72%**	**165.65%**

The table also contains results for database *LOB1through6.icradb-v10*, model *good2* and forecast *cons21u*. LOB4 is characterised by a volatile model, the aggregate LOB1-6 is less volatile. The row $BSCR$ depicts the result based on QIS5 Method 2, and the row SCR (simulations) depicts the simulations based internal model result.

The QIS5 Method 2 result tends to be close to the simulation-based result for less volatile models. The more volatile the model, the higher the divergence is.

10 Conclusion

Solvency II risk metrics have a nominal nature, and practitioners have the freedom to pick up an estimation method, constrained by supervisory approval.

Nevertheless, Solvency II risk metrics are based on statistical inference. Munroe et al. (2015) argue in the Conclusion that sound inference presupposes an accurate modelling framework. Where simplifications and artificial assumptions are minimal, the inference can be expected to be more sound. We refer to the proportionality principle and Chain Ladder method in the

QIS5 standard formula for reserve risk, and to inherent simplifications in reserve risk in the context of QIS5 standard formula mentioned above (section 7).

As follows from the case study, the simulation-based internal model (the SII module from ICRFS-Plus) is likely to produce a similar or lower estimate for SCR, comparing with $BSCR$ from the standard formula method 2 for reserve risk (taken from the PTF forecast summary in ICRFS-Plus).

The simulation-based internal model SCR estimate is a conservative estimate. This is because the simulation algorithm uses the unconditional value at risk $VaR_{99.5\%}(L_p)$ for $p > 1$, not $E_{L_1}(VaR_{99.5\%}(L_p)|L_1)$, due to additional simulation complexity. Research to quantify the conservativeness of the estimate is in progress.

11 Appendix A. QIS5 Standard Formula for SCR

Reserve risk and premium risk parameters are the inputs into the total SCR standard formula (see European Commission et al. (2010)). There is a standard formula for reserve risk (SCR.9.29) and for premium risk (SCR.9.25), where the standard deviation is set to be proportional to the mean. There is also a standard formula with *undertaking-specific* parameters in *Reserve Risk* that we are focusing on.

The Method 2 in the standard formula with *undertaking-specific* parameters in *Reserve Risk* is known as *Merz-Wüthrich* approach (Merz and Wüthrich (2008)). We will cite the following key paragraphs from European Commission et al. (2010) below.

> SCR.10.37. Under the Merz-Wüthrich approach used in methods 2 and 3 below, the estimator explicitly only captures the prediction error and does not capture model error (for example the chain ladder assumptions do not hold) or the error in case the past data do not reflect the future business.
>
> SCR.10.48. This approach is based on the mean squared error of prediction of the claims development result over the one year and fitting a model to these results. The mean squared errors are calculated using the approach detailed in "Modelling The Claims Development Result For Solvency Purposes" by Michael Merz

and Mario V Wüthrich, Casualty Actuarial Society E-Forum, Fall 2008.

The following notation is used in QIS5,

- $MSEP$ - mean square error of prediction of $CDR(1)$, see SCR.10.48. ,
- PCO - provision for claims outstanding,
- $\sigma_{u,res,lob}$ - undertaking standard deviation for reserve risk for LoB,
- $\sigma_{res,lob}$ - standard deviation for reserve risk for LoB,
- σ_{lob} - standard deviation for LoB risk,
- σ - overall standard deviation,
- NL_{pr} - capital requirement for premium and reserve risk,
- SCR_{nl} - non-life SCR,
- $BSCR$ - basic SCR.

See figure 2 for QIS5 SCR standard formula flowchart. The diagram shows how undertaking specific parameters obtained from ICRFS-Plus (either from the QIS5 method 2 standard formula in reserve risk or from one of partial internal models) can be plugged into the QIS5 SCR standard formula.

We will cite paragraphs from European Commission et al. (2010) referred in the figure 2.

> SCR.10.50. Therefore $\sigma_{u,res,lob} = \sqrt{MSEP}/PCO_{lob}$
> SCR.10.9. Undertaking should derive the undertaking-specific parameters as follows:
> ...
> **For reserve risk:**
> Undertakings should derive new parameters as follows:
> $\sigma_{res,lob} = c\ \sigma_{u,res,lob} + (1-c)\sigma_{M,res,lob}$
> Where
> c = credibility factor as defined in SCR.10.10. $c = 1$ for time series that are long enough,

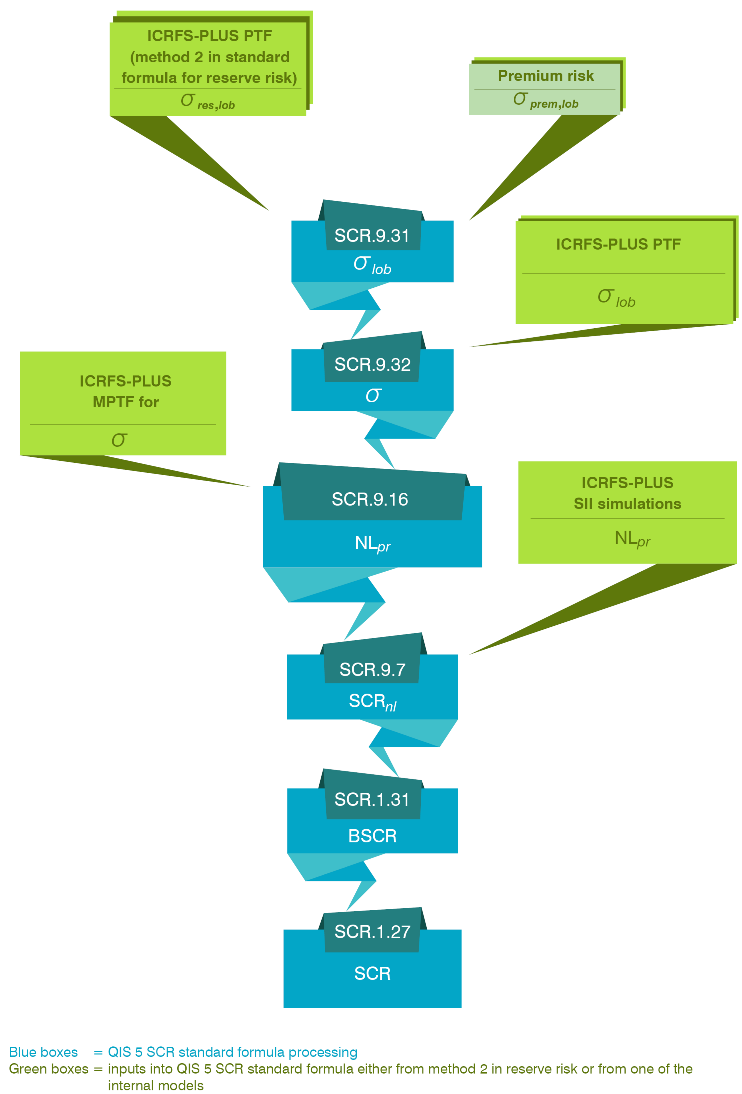

Figure 2: QIS5 SCR standard formula flowchart and ICRFS-Plus

$\sigma_{u,res,lob}$ = undertaking-specific estimate of the standard deviation for reserve risk,

$\sigma_{M,res,lob}$ = standard parameters of the standard deviation for reserve risk which are provided in SCR.9 (Non Life Underwriting Risk Section).

SCR.9.31. The standard deviation for premium and reserve risk in the individual LoB is defined by aggregating the standard deviations for both subrisks under the assumption of a correlation coefficient of $\alpha = 0.5$:

$$\sigma_{lob} = \frac{\sqrt{\begin{array}{c}(\sigma_{prem,lob} \cdot V_{prem,lob})^2 + \\ 2\alpha \cdot \sigma_{prem,lob} \cdot \sigma_{res,lob} \cdot V_{prem,lob} \cdot V_{res,lob} + \\ (\sigma_{res,lob} \cdot V_{res,lob})^2\end{array}}}{V_{prem,lob}+V_{res,lob}}$$

SCR.9.32. The overall standard deviation σ is determined as follows:

$$\sigma = \sqrt{\frac{1}{V^2}\sum_{r,c} CorrLoB_{r,c} \cdot \sigma_r \cdot \sigma_c \cdot V_r \cdot V_c}$$

where

r, c = All indices of the form (lob)

$CorrLoB_{r,c}$ = The entries of the correlation matrix CorrLob

V_r, V_c = Volume measures for the individual lines of business, as defined in step 1

SCR.9.16. The capital requirement for the combined premium risk and reserve risk is determined as follows:

$NL_{pr} = \rho(\sigma) \cdot V$

where

V = Volume measure

σ = Combined standard deviation

$\rho(\sigma)$ = A function of the combined standard deviation

SCR.9.18. The function $\rho(\sigma)$ is set such that, assuming a lognormal distribution of the underlying risk, a risk capital requirement consistent with the $VaR_{99.5\%}$ calibration objective is produced. Roughly, $\rho(\sigma) \approx 3\sigma$

SCR.9.28. The volume measure for reserve risk for each individual LoB is determined as follows:

$V_{res,lob} = PCO_{lob}$

SCR.10.13. The [premium risk] analysis should be performed using the net earned premiums as the volume measure. See also

SCR.9.23.

SCR.9.7. The capital requirement for non-life underwriting risk is derived by combining the capital requirements for the non-life sub-risks using a correlation matrix as follows:

$SCR_{nl} = \sqrt{\sum CorrNL_{r,c} \cdot NL_r \cdot NL_c}$

where

$CorrNL_{r,c}$ = The entries of the correlation matrix $CorrNL$

NL_r, NL_c = Capital requirements for individual non-life underwriting sub-risks according to the rows and columns of correlation matrix $CorrNL$

and where the correlation matrix CorrNL is defined as:

$CorrNL$	NL_{pr}	NL_{lapse}	NL_{CAT}
NL_{pr}	1		
NL_{lapse}	0	1	
NL_{CAT}	0.25	0	1

SCR.1.31. The $BSCR$ is determined as follows:

$$BSCR = \sqrt{\sum_{i,j} Corr_{i,j} \cdot SCR_i \cdot SCR_j} + SCR_{intangibles}$$

where

$Corr_{i,j}$ = the entries of the correlation matrix $Corr$

SCR_i, SCR_j = Capital requirements for the individual SCR risks according to the rows and columns of the correlation matrix $Corr$

$SCR_{intangibles}$ = the capital requirement for intangible asset risk calculated in accordance with SCR.4

SCR.1.27. The SCR is determined as follows:

$SCR = BSCR + Adj + SCR_{Op}$

12 Appendix B. Conditional statistics

Let $X_{1..p} \sim N(\mu_{1..p}, \Sigma)$ be a p-variate normally distributed random variable. Then $Y_{1..p} = \exp(X_{1..p})$ is a p-variate lognormally distributed, $Y_{1..p} \sim LN(\mu_{1..p}, \Sigma)$. From the properties of lognormal distribution,

$$E(Y_p) = \exp(\mu_p + \sigma_p^2/2) \quad (12)$$

$$Var(Y_p) = (E(Y_p))^2 \cdot \left(\exp(\sigma_p^2) - 1\right) \quad (13)$$

where $\sigma_p^2 = \Sigma_{pp}$.

Consider partitioning of Σ at the p-th row and column,

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{1p} \\ \Sigma_{p1} & \Sigma_{pp} \end{pmatrix}.$$

From the properties of conditional normal distribution

$$X_p|X_{1..p-1} \sim N(\mu_p + \Sigma_{p1}\Sigma_{11}^{-1}(x_{1..p-1} - \mu_{1..p-1}), \Sigma_{pp|1..p-1}) \tag{14}$$

and

$$Y_p|X_{1..p-1} \sim LN(\mu_p + \Sigma_{p1}\Sigma_{11}^{-1}(x_{1..p-1} - \mu_{1..p-1}), \Sigma_{pp|1..p-1}) \tag{15}$$

where

$$\Sigma_{pp|1..p-1} = \Sigma_{pp} - \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p} \tag{16}$$

is a short form for $\Sigma_{pp}|(X_1..X_{p-1})$.

By the law of *iterated expectations*

$$E_{1..p-1}(E(Y_p|X_{1..p-1})) = E(Y_p).$$

Since $Y_p|X_{1..p-1}$ is lognormal,

$$Var(Y_p|X_{1..p-1}) = (E(Y_p|X_{1..p-1}))^2 \cdot \left(\exp(\Sigma_{pp} - \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}) - 1\right). \tag{17}$$

To find expectation of conditional variance $E_{1..p-1}(Var(Y_p|X_{1..p-1}))$ notice the second term on the right side of the above identity $(\exp(\Sigma_{pp} - \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}) - 1)$ is a constant. Substituting conditional mean and variance (15) into (12) and raising the result in the power of 2, the first term yields

$$(E(Y_p|X_{1..p-1}))^2 = \exp(2(\mu_p + \Sigma_{p1}\Sigma_{11}^{-1}(x_{1..p-1} - \mu_{1..p-1}) + \Sigma_{pp} - \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}).$$

As long as $x_{1..p-1}$ is a multivariate normal, $(E(Y_p|X_{1..p-1}))^2$ is a lognormal random variable, $(E(Y_p|X_{1..p-1}))^2 \sim LN\left(2\mu_p + \Sigma_{pp} - \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}, 4\Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}\right)$. Then from (12)

$$E_{1..p-1}((E(Y_p|X_{1..p-1}))^2) = \exp\left(2\mu_p + \Sigma_{pp} + \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}\right).$$

Finally, taking expectation from the both sides of (17) and substituting the above result,

$$\begin{aligned} E_{1..p-1}(Var(Y_p|X_{1..p-1})) &= \exp\left(2\mu_p + \Sigma_{pp} + \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}\right) \\ &\quad \left(\exp(\Sigma_{pp} - \Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}) - 1\right). \end{aligned}$$

Rearranging the right hand side of the above,

$$E_{1..p-1}(Var(Y_p|X_{1..p-1})) = \exp\left(2\mu_p + \Sigma_{pp}\right)(\exp(\Sigma_{pp}) - \exp(\Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p})).$$

Further, using (12) to substitute $\exp\left(2\mu_p + \Sigma_{pp}\right)$ with $(E(Y_p))^2$ and using (16) to substitute $\Sigma_{p1}\Sigma_{11}^{-1}\Sigma_{1p}$ with $(\Sigma_{pp} - \Sigma_{pp|1..p-1})$, the above yields the expectation of conditional variance

$$E_{1..p-1}(Var(Y_p|X_{1..p-1})) = (E(Y_p))^2 \cdot \left(\exp(\Sigma_{pp}) - \exp(\Sigma_{pp} - \Sigma_{pp|1..p-1})\right). \quad (18)$$

Corollary 1. *Expectation of conditional covariance.*

Expanding Σ by adding the $(p+1)$-th row and column, and by the same arguments and similar expressions as for the expectation of conditional variance

$$\begin{aligned} E_{1..p-1}(Cov(Y_p, Y_{p+1}|X_{1..p-1})) &= E(Y_p)E(Y_{p+1}) \\ &\quad \left(\exp(\Sigma_{p,(p+1)}) - \exp(\Sigma_{p,(p+1)} - \Sigma_{p,(p+1)|1..p-1})\right). \end{aligned} \quad (19)$$

Corollary 2. *Variance of conditional expectation.*

From the low of *total variance*,

$$Var(E_{1..p-1}(Y_p|X_{1..p-1})) = Var(Y_p) - E_{1..p-1}(Var(Y_p|X_{1..p-1})).$$

By substituting (13) and (18) in the above,

$$Var(E_{1..p-1}(Y_p|X_{1..p-1})) = (E(Y_p))^2 \cdot \left(\exp(\Sigma_{pp} - \Sigma_{pp|1..p-1}) - 1\right).$$

Corollary 3. *Generalization for log-scale observations X as a linear combination of parameters (state space models).*

Denote a $m * m$ covariance of parameters matrix as S and (future observations) design matrix $T * m$ as H, where m is the number of parameters and T is the number of future observations. Then by properties of variance,

$$\Sigma = HSH^T + \Omega$$

where Ω is $T * T$ process covariance matrix, diagonal in PTF context.

In the ICRFS context, the conditioning is on the first future diagonal C, and T is the number of future observations that excludes the set C. Then

$$\Sigma|C = H(S|C)H^T + \Omega$$

and (18) and (19) are transformed as

$$\begin{aligned} E(Var(Y_p|C)) &= (E(Y_p))^2 \cdot \big(\exp(\Sigma_{pp}) - \exp(\Sigma_{pp} - \Sigma_{pp|C})\big) \\ E(Cov(Y_p|C, Y_q|C)) &= E(Y_p)E(Y_q)\,\big(\exp(\Sigma_{pq}) - \exp(\Sigma_{pq} - \Sigma_{pq|C})\big) \end{aligned}$$

where p and q are future observations outside of the set C. Covariances of all combinations of (p, q) are required in order to calculate a variance of the sum of the future observations after the first future calendar period. Obviously, $S|C$ is the prerequisite for such calculations.

13 Appendix C. Case Study calculation details

The following steps are performed to calculate $BSCR$ from

$$PCO = 2,298,680; S.D.(CDR) = 106,916.$$

1. $$\sigma = \frac{S.D.(CDR)}{PCO} = \frac{106,916}{2,298,680} = 0.0465.$$

2. Use method of moments to approximate a distribution with mean $\mu = 1$ and standard deviation $\sigma = 0.0465$ with a lognormal distribution $LN(m, s)$ by solving the system of equitions

$$\begin{aligned} \exp(m + s^2/2) &= \mu, \\ \mu^2 \cdot \big(\exp(s^2) - 1\big) &= \sigma^2. \end{aligned}$$

   The solution is

$$s^2 = \ln\left(1 + \frac{\sigma^2}{\mu^2}\right), \quad (20)$$

$$m = \ln\mu - s^2/2. \quad (21)$$

Substituting μ and σ in the above,

$$\begin{aligned} s &= 0.04648, \\ m &= -0.001. \end{aligned}$$

As follows from the Taylor series expansion of $\ln\left(a + \frac{\sigma^2}{\mu^2}\right)$ from (20) at a point $a = 1$ where $\mu = 1$ and σ is small, $s \approx \sigma$. From (21), with $\mu = 1$ and a small s, $m \approx 0$.

3. The 99.5% quantile of the normal distribution $N(m, s)$ is $m + s \cdot \Phi^{-1}(0.995) = 0.1187$, where $\Phi^{-1}(0.995) \approx 2.576$

4. The 99.5% quantile on lognormal scale is the exponent of the above, $exp(0.1187) = 1.1260$.

5. The 99.5% quantile on lognormal scale above the mean $\mu = 1$ is 0.1260. This is $\rho(\sigma)$, see Appendix A, SCR.9.18. Note, $3s = 0.1395$ is a rough approximation of $\rho(\sigma)$ ($2.576 \cdot s = 0.1197$ is another rough approximation).

6. Scaling the above by the volume measure PCO produces capital requirement $2,298,680 \cdot 0.1260 = \$289,608$.

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